Computers and Mathematics with Applications 61 (2011) 790–796

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Fixed point theorems for weakly C-contractive mappings in ordered metric spaces ☆

J. Harjani, B. López, K. Sadarangani *

Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain

ARTICLE INFO

Article history:
Received 5 April 2010
Received in revised form 14 December 2010
Accepted 14 December 2010

This paper is dedicated to Professor José Manuel Méndez Pérez on the occasion of his 60th birthday.

Keywords:
Fixed point
Ordered metric space
Weak C-contraction

ABSTRACT

The purpose of this paper is to present some fixed point results for weakly C-contractive mappings in a complete metric space endowed with a partial order.

1. Introduction

The Banach contraction mapping is one of the pivotal results of functional analysis. It is widely considered as the source of metric fixed point theory. Also its significance lies in its vast applicability in a number of branches of mathematics.

Generalization of the above principle has been a extensively investigated branch of research. In particular, Chatterjea in [1] introduced the following definition.

Definition 1.1. A mapping $T: X \to X$ where (X, d) is a metric space is said to be a C-contraction if there exists $\alpha \in \left(0, \frac{1}{2}\right)$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx)).$$

Chatterjea in [1] proved that if X is complete, then every C-contraction has a unique fixed point. In establishing this result there is no requirement of continuity of the C-contraction.

Choudhury in [2] introduced a generalization of C-contraction given by the following definition.

Definition 1.2. A mapping $T: X \to X$, where (X, d) is a metric space is said to be weakly C-contractive (or a weak C-contraction) if for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \varphi(d(x, Ty), d(y, Tx)),$$

where $\varphi: [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only $x = y = 0$.

☆ This research was partially supported by "Ministerio de Educación y Ciencia", Project MTM 2007/65706.

* Corresponding author.
E-mail addresses: matematicas.jackie@yahoo.es (J. Harjani), blopez@dma.ulpgc.es (B. López), ksadaran@dma.ulpgc.es (K. Sadarangani).

0898-1221/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.12.027

In [2] the author proves that if X is complete then every weak C-contraction has a unique fixed point.

The purpose of this paper is to present this last result in the context of ordered metric spaces.

Existence of fixed point in partially ordered sets has been considered recently in [3–19]. Tarski's theorem is used in [11] to show the existence of solutions for fuzzy equations and in [13] to prove existence theorems for fuzzy differential equations. In [8,9,12,15,18] some applications to ordinary differential equations and to matrix equations are presented, respectively. In [5,7,19] some fixed point theorems are proved for a mixed monotone mapping in a metric space endowed with a partial order and the authors applied their results to problems of existence and uniqueness of solutions for some boundary value problems.

In the context of ordered metric spaces, the usual contraction is weakened but at the expense that the operator is monotone. The main idea in [12,18] involve combining the ideas in the contraction principle with those in the monotone interactive technique [20].

2. Fixed point results: nondecreasing case

Our starting point is the following definition.

Definition 2.1. If $(X, \leq)$ is a partially ordered set and $T: X \to X$ we say that T is monotone nondecreasing if, for $x, y \in X$,

$$x \leq y \Rightarrow Tx \leq Ty.$$

This definition coincides with the notion of a nondecreasing function in the case where $X = \mathbb{R}$ and $\leq$ represents the usual total order in $\mathbb{R}$.

In what follows, we present the following theorem which is a version of the result in [2] in the context of ordered metric spaces when the operator is nondecreasing.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T: X \to X$ be a continuous and nondecreasing mapping such that*

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \varphi(d(x, Ty), d(y, Tx)), \quad \text{for } x \geq y \tag{1}$$

where $\varphi: [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only is $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$ then T has a fixed point.

Proof. If $Tx_0 = x_0$ then the proof is finished. Suppose that $x_0 < Tx_0$.

Since $x_0 < Tx_0$ and T is a nondecreasing mapping, we obtain by induction that

$$x_0 < Tx_0 \leq T^2x_0 \leq T^3x_0 \leq \cdots \leq T^nx_0 \leq T^{n+1}x_0 \leq \cdots.$$

Put $x_{n+1} = Tx_n$. Then, for each integer $n \geq 1$, from (1) and, as the elements x_{n-1} and x_n are comparable, we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \frac{1}{2}(d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)) - \varphi(d(x_n, Tx_{n-1}), d(x_{n-1}, Tx_n)) \\ &= \frac{1}{2}(d(x_n, x_n) + d(x_{n-1}, x_{n+1})) - \varphi(d(x_n, x_n), d(x_{n-1}, x_{n+1})) \\ &= \frac{1}{2}(d(x_{n-1}, x_{n+1})) - \varphi(0, d(x_{n-1}, x_{n+1})) \\ &\leq \frac{1}{2}(d(x_{n-1}, x_{n+1})) \\ &\leq \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})). \end{aligned} \tag{2}$$

The last inequality gives us

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}).$$

Thus $(d(x_{n+1}, x_n))$ is a decreasing sequence of nonnegative real numbers and hence it is convergent.

Let

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{3}$$

Letting $n \to \infty$ in (2) we have

$$r \leq \lim_{n\to\infty} \frac{1}{2} d(x_{n-1}, x_{n+1}) \leq \frac{1}{2}(r + r) = r$$

or, equivalently,

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r. \tag{4}$$

Again, making $n \to \infty$ in (2) and using (3), (4) and the continuity of φ we obtain

$$r \le \frac{1}{2}2r - \varphi(0, 2r) = r - \varphi(0, 2r) \le r$$

and, consequently, $\varphi(0, 2r) = 0$. This gives us that $r = 0$ by our assumption about φ.

Thus we have that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{5}$$

In what follows, we will prove that (x_n) is a Cauchy sequence.

If otherwise, then there exists $\epsilon > 0$ for which we can find subsequences $(x_{m(k)})$ and $(x_{n(k)})$ of (x_n) with $n(k) > m(k) > k$ such that for every k

$$d(x_{n(k)}, x_{m(k)}) \ge \epsilon. \tag{6}$$

Further, corresponding to $m(k)$ we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (6).

Then

$$d(x_{n(k)-1}, x_{m(k)}) < \epsilon. \tag{7}$$

Using (6), (7) and the triangular inequality, we have

$$\begin{aligned} \epsilon &\le d(x_{n(k)}, x_{m(k)}) \\ &\le d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\ &< d(x_{n(k)}, x_{n(k)-1}) + \epsilon. \end{aligned}$$

Making $k \to \infty$ in the above inequality and using (5)

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \epsilon. \tag{8}$$

Again, the triangular inequality gives us

$$\begin{aligned} &d(x_{m(k)}, x_{n(k)-1}) \le d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1}). \\ &d(x_{m(k)-1}, x_{n(k)}) \le d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}). \end{aligned}$$

Letting $k \to \infty$ in the above two inequalities and using (5) and (8) we get

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon. \tag{9}$$

As $n(k) > m(k)$ and $x_{n(k)-1}$ and $x_{m(k)-1}$ are comparable, using (1) we have

$$\begin{aligned} \epsilon &\le d(x_{n(k)}, x_{m(k)}) \\ &= d(Tx_{n(k)-1}, Tx_{m(k)-1}) \\ &\le \frac{1}{2}(d(x_{n(k)-1}, Tx_{m(k)-1}) + d(x_{m(k)-1}, Tx_{n(k)-1})) - \varphi(d(x_{n(k)-1}, Tx_{m(k)-1}), d(x_{m(k)-1}, Tx_{n(k)-1})) \\ &= \frac{1}{2}(d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)})) - \varphi(d(x_{n(k)-1}, x_{m(k)}), d(x_{m(k)-1}, x_{n(k)})). \end{aligned}$$

Making $k \to \infty$ and taking into account (8), (9) and the continuity of φ, we have

$$\epsilon \le \frac{1}{2}(\epsilon + \epsilon) - \varphi(\epsilon, \epsilon) \le \epsilon$$

and from the last inequality $\varphi(\epsilon, \epsilon) = 0$. By our assumption about φ, we have $\epsilon = 0$ which is a contradiction.

This proves that (x_n) is a Cauchy sequence. Since X is a complete metric space, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z.$$

Moreover, the continuity of T implies that

$$z = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = Tz$$

and this proves that z is a fixed point for T. $\square$

In what follows we prove that Theorem 2.1 is still valid for T not necessarily continuous, assuming the following hypothesis in X (which appears in Theorem 1 of [12]):

$$\text{if } (x_n) \text{ is a nondecreasing sequence in X such that } x_n \to x \text{ then } x_n \le x \text{ for all } n \in \mathbb{N}. \tag{10}$$

Theorem 2.2. *Let $(X, \le)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies (10). Let $T: X \to X$ be a nondecreasing mapping such that*

$$d(Tx, Ty) \le \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \varphi(d(x, Ty), d(y, Tx)), \quad \text{for } x \ge y,$$

where $\varphi: [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \le Tx_0$ then T has a fixed point.

Proof. Following the proof of Theorem 2.1 we only have to check that $Tz = z$. As (x_n) is a nondecreasing sequence in X and $x_n \to z$, then, the condition (10) gives us that $x_n \le z$ for every $n \in \mathbb{N}$ and, consequently, the contractive condition (1) gives us

$$\begin{aligned} d(x_{n+1}, Tz) &= d(Tx_n, Tz) \\ &\le \frac{1}{2}(d(x_n, Tz) + d(z, Tx_n)) - \varphi(d(x_n, Tz), d(z, Tx_n)) \\ &= \frac{1}{2}(d(x_n, Tz) + d(z, x_{n+1})) - \varphi(d(x_n, Tz), d(z, x_{n+1})). \end{aligned}$$

Letting $n \to \infty$ and using the continuity of φ we have

$$\begin{aligned} d(z, Tz) &\le \frac{1}{2}(d(z, Tz) + d(z, z)) - \varphi(d(z, Tz), d(z, z)) \\ &= \frac{1}{2}(d(z, Tz)) - \varphi(d(z, Tz), 0) \\ &\le \frac{1}{2}d(z, Tz) \end{aligned}$$

and this is a contraction unless $d(z, Tz) = 0$, or, equivalently, $Tz = z$.

This completes the proof. □

Now, we present an example where it can be appreciated that hypotheses in Theorems 2.1 and 2.2 do not guarantee uniqueness of the fixed point. This example appears in [12].

Let $X = \{(1, 0), (0, 1)\} \subset \mathbb{R}^2$ and consider the usual order

$$(x, y) \le (z, t) \Leftrightarrow x \le z \text{ and } y \le t.$$

Thus, $(X, \le)$ is a partially ordered set whose different elements are not comparable. Besides, (X, d_2) is a complete metric space considering d_2 the Euclidean distance. The identity map $T(x, y) = (x, y)$ is trivially continuous and nondecreasing and condition (1) of Theorem 2.1 is satisfied since elements in X are only comparable to themselves. Moreover, $(1, 0) \le T(1, 0) = (1, 0)$ and T has two fixed points in X.

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorems 2.1 and 2.2. This condition is (and it appears in [18]):

for $x, y \in X$ there exists a lower bound or an upper bound.

In [12] it is proved that the above mentioned condition is equivalent to:

$$\text{for } x, y \in X \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{11}$$

Theorem 2.3. *Adding condition (11) to the hypotheses of Theorem 2.1 (or Theorem 2.2) we obtain the uniqueness of the fixed point of T.*

Proof. Suppose that there exist $z, y \in X$ which are fixed points of T. We distinguish two cases:

Case 1. If y is comparable to z then $T^n y = y$ is comparable to $T^n z = z$ for $n = 1, 2, \ldots$, and

$$\begin{aligned} d(y, z) &= d(T^n y, T^n z) \\ &\le \frac{1}{2}(d(T^{n-1}y, T^n z) + d(T^{n-1}z, T^n y)) - \varphi(d(T^{n-1}y, T^n z), d(T^{n-1}z, T^n y)) \\ &= \frac{1}{2}(d(y, z) + d(z, y)) - \varphi(d(y, z), d(z, y)) \\ &= d(y, z) - \varphi(d(y, z), d(z, y)) \\ &\le d(y, z) \end{aligned}$$

and this inequality gives us $\varphi(d(y,z), d(z,y)) = 0$, and, by our assumption about φ, $d(y,z) = 0$, or, equivalently, $y = z$.

Case 2. If y is not comparable to z then there exists $x \in X$ comparable to y and z. Monotonicity of T implies that $T^n x$ is comparable to $T^n y = y$ and to $T^n z = z$ for $n = 1, 2, \ldots$. Condition (1) of Theorem 2.1 gives us

$$\begin{aligned} d(z, T^n x) &= d(T^n z, T^n x) \\ &\le \frac{1}{2}(d(T^{n-1}z, T^n x) + d(T^{n-1}x, T^n z)) - \varphi(d(T^{n-1}z, T^n x), d(T^{n-1}x, T^n z)) \\ &= \frac{1}{2}(d(z, T^n x) + d(T^{n-1}x, z)) - \varphi(d(z, T^n x), d(T^{n-1}x, z)) \\ &\le \frac{1}{2}(d(z, T^n x) + d(z, T^{n-1}x)) \end{aligned} \tag{12}$$

and from the above inequality we get

$$d(z, T^n x) \le d(z, T^{n-1}x).$$

This proves that the nonnegative decreasing sequence $(d(z, T^n x))$ is convergent. Put $\lim_{n\to\infty} d(z, T^n x) = r$.

Letting $n \to \infty$ in (12) and taking into account the continuity of φ we obtain

$$r \le \frac{1}{2}(r + r) - \varphi(r, r) \le r.$$

This gives us $\varphi(r, r) = 0$, and, by our assumption about φ, $r = 0$.
Consequently, $\lim_{n\to\infty} d(z, T^n x) = 0$.
Analogously, it can be proved that $\lim_{n\to\infty} d(y, T^n x) = 0$.
Finally, the uniqueness of the limit gives us $y = z$.
This finishes the proof. □

Remark 2.1. Notice that if $(X, \le)$ is a totally ordered set, condition (11) is obviously satisfied and we obtain uniqueness of the fixed point.

Remark 2.2. Suppose that $\alpha \in \left(0, \frac{1}{2}\right)$. If we consider in Theorem 2.1 (or in Theorems 2.2 and 2.3) as φ the function $\varphi: [0,\infty)^2 \to [0,\infty)$ defined by

$$\varphi(a, b) = \left(\frac{1}{2} - \alpha\right)(a + b)$$

which, obviously, satisfies that $\varphi(a, b) = 0$ if and only if $a = b = 0$, condition (1) of Theorem 2.1 can be rewritten as

$$d(Tx, Ty) \le \alpha(d(x, Ty) + d(y, Tx)) \quad \text{for } x \ge y$$

and Theorem 2.1 (or 2.2 or 2.3) can be considered as the version in the context of ordered metric spaces of the result proved by Chatterjea in [1] for nondecreasing mappings.

3. Fixed point results: nonincreasing case

In this section we present a fixed point theorem for weakly C-contractive mappings when the operator T is nonincreasing. We start with the following definition.

Definition 3.1. If $(X, \le)$ is a partially ordered set and $T: X \to X$ we say that T is monotone nonincreasing if for $x, y \in X$

$$x \le y \Rightarrow Tx \ge Ty.$$

The main result of this section is the following theorem.

Theorem 3.1. *Let $(X, \le)$ be a partially ordered set satisfying condition (11) and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T: X \to X$ be a nonincreasing mapping such that*

$$d(Tx, Ty) \le \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \varphi(d(x, Ty), d(y, Tx)), \quad \text{for } x \ge y, \tag{13}$$

where $\varphi: [0,\infty)^2 \to [0,\infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$.
If there exists $x_0 \in X$ with $x_0 \le Tx_0$ or $x_0 \ge Tx_0$ then $\inf\{d(x, Tx): x \in X\} = 0$.
If, in addition, X is compact and T is continuous, then T has a unique fixed point.

Proof. If $Tx_0 = x_0$ then it is obvious that $\inf\{d(x, Tx): x \in X\} = 0$.

Suppose that $x_0 < Tx_0$ (the same argument serves for $x_0 < Tx_0$).

In virtue that T is nonincreasing the consecutive terms of the sequence $(T^n x_0)$ are comparable and using (13) we can obtain

$$\begin{aligned} d(T^{n+1}x_0, T^n x_0) &\le \frac{1}{2}(d(T^n x_0, T^n x_0) + d(T^{n-1}x_0, T^{n+1}x_0)) - \varphi(d(T^n x_0, T^n x_0), d(T^{n-1}x_0, T^{n+1}x_0)) \\ &= \frac{1}{2}(d(T^{n-1}x_0, T^{n+1}x_0)) - \varphi(0, d(T^{n-1}x_0, T^{n+1}x_0)) \\ &\le \frac{1}{2}(d(T^{n-1}x_0, T^n x_0) + d(T^n x_0, T^{n+1}x_0)). \end{aligned}$$

From this inequality we have that $(d(T^{n+1}x_0, T^n x_0))$ is a nonnegative decreasing sequence with limit $r \ge 0$. Using a similar argument that in Theorem 2.1 we can prove that $r = 0$.

This means that $\lim_{n\to\infty} d(T^{n+1}x_0, T^n x_0) = 0$ and, consequently, $\inf\{d(x, Tx): x \in X\} = 0$.

This finishes the first part of the proof of our theorem.

Now, suppose that X is compact and T is continuous.

Taking into account that the mapping

$$\begin{aligned} &X \longrightarrow \mathbb{R}^+ \\ &x \longrightarrow d(x, Tx), \end{aligned}$$

is continuous (note that this mapping can be obtained as

$$\begin{aligned} &X \longrightarrow X \times X \longrightarrow \mathbb{R}^+ \\ &x \longrightarrow (x, Tx) \longrightarrow d(x, Tx), \end{aligned}$$

and, obviously, this composition of mappings is continuous because T is continuous), and since X is compact, we can find $z \in X$ such that

$$d(z, Tz) = \inf\{d(x, Tx): x \in X\}.$$

Taking into account the first part of the theorem

$$d(z, Tz) = 0$$

and, therefore, z is a fixed pint of T.

The uniqueness of the fixed point is proved as in Theorem 2.3. $\square$

Remark 3.1. A parallel result in the nonincreasing case cannot be obtained using a same reasoning that in Theorem 2.1, because the proof of Cauchy character of the sequence (x_n) fails since $x_{n(k)-1}$ and $x_{m(k)-1}$ can be not comparable if T is a nonincreasing operator.

4. Examples

In this section we present some examples which illustrate our results.

Example 4.1. Let $X = \{(0, 1), (1, 0), (1, 1)\} \subset \mathbb{R}^2$ with the euclidean distance d_2. (X, d_2) is, obviously, a complete metric space. Moreover, we consider the order $\le$ in X given by $R = \{(x, x): x \in X\}$.

Notice that the elements in X are only comparable to themselves.

Also we consider $T: X \to X$ given by

$$\begin{aligned} &T(1, 0) = (0, 1) \\ &T(0, 1) = (1, 0) \\ &T(1, 1) = (1, 1). \end{aligned}$$

Obviously, T is a continuous and nondecreasing mapping, and, moreover, $(1, 1) \le T(1, 1)$. As the elements in X are only comparable to themselves, condition (1) appearing in Theorem 2.1 is, obviously, satisfied. Finally, Theorem 2.1 gives us the existence of a fixed point for T (which it is obviously the point $(1, 1)$).

On the other hand,

$$\sqrt{2} = d_2(T(1, 0), T(0, 1)) = d_2((0, 1), (1, 0))$$

and

$$\frac{1}{2}(d_2((1, 0), T(0, 1)) + d_2((0, 1), T(1, 0))) = 0$$

and this proves that the operator T is not a weak C-contraction (see Definition 2.1) and, consequently, this example cannot be treated by the main result of [2].

Notice that in this example we obtain uniqueness of the fixed point and condition (11) appearing in Theorem 2.3 is not satisfied by $(X, \leq)$ (precisely, for the elements $(0, 1), (1, 0) \in X$).

This proves that condition (11) is not a necessary condition for the uniqueness of the fixed point.

Example 4.2. Consider the same space X that in Example 4.1 with the euclidean distance d_2 and with the order given by $R = \{(x, x): x \in X\} \cup \{((0, 1), (1, 1))\}$.

Let T be the operator $T: X \to X$ defined by $T(0, 1) = (0, 1)$, $T(1, 1) = (0, 1)$ and $T(1, 0) = (1, 0)$.

Obviously, T is a continuous and nondecreasing mapping since $(0, 1) \leq (1, 1)$ and $T(0, 1) = (0, 1) \leq T(1, 1) = (0, 1)$.

In what follows, we prove that T satisfies condition (1) appearing in Theorem 2.1.

In fact, for $(0, 1) \leq (1, 1)$

$$d(T(0, 1), T(1, 1)) = d((0, 1), (0, 1)) = 0,$$

and, consequently, condition (1) is satisfied.

As $(0, 1) \leq T(0, 1)$, Theorem 2.1 says us that T has a fixed point (in this case, $(0, 1)$ and $(1, 0)$ are fixed points of T).

Notice that, in this case, we have not uniqueness of the fixed point. Moreover, $(X, \leq)$ does not satisfy condition (11) of Theorem 2.3.

On the other hand, the operator T is not a weak C-contraction because

$$d_2(T(1, 0), T(0, 1)) = d_2((1, 0), (0, 1)) = \sqrt{2},$$

and

$$\begin{aligned}
&\frac{1}{2}(d_2((1, 0), T(0, 1)) + d_2((0, 1), T(1, 0))) - \varphi(d_2((1, 0), T(0, 1)), d_2((0, 1), T(1, 0))) \\
&\quad = \frac{1}{2}(d_2((1, 0), (0, 1)) + d_2((0, 1), (1, 0))) - \varphi(d_2((1, 0), (0, 1)), d_2((0, 1), (1, 0))) \\
&\quad = \frac{1}{2}(\sqrt{2} + \sqrt{2}) - \varphi(\sqrt{2}, \sqrt{2}) \\
&\quad = \sqrt{2} - \varphi(\sqrt{2}, \sqrt{2}) \\
&\quad < \sqrt{2},
\end{aligned}$$

because $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Thus, this example cannot be studied by the main result of [2] (Theorem 2.1 of [2]).

Acknowledgements

The authors express their sincere thanks to the reviewers for their valuable suggestions in improving the paper.

References

[1] S.K. Chatterjea, Fixed point theorems, C. R. Acad. Bulgare Sci. 25 (1972) 727–730.
[2] B.S. Choudhury, Unique fixed point theorem for weak C-contractive mappings, Kathmandu University Journal of Science, Engineering and Technology 5 (1) (2009) 6–13.
[3] R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, Appl. Anal. 87 (2008) 109–116.
[4] I. Altun, H. Simsek, Some fixed point theorems on ordered metric spaces and application, Fixed Point Theory Appl. (2010) Article ID 621469.
[5] Dz. Burgic, S. Kalabusic, M.R.S. Kulenovic, Global attractivity results for mixed monotone mappings in partially ordered complete metric spaces, Fixed Point Theory Appl. (2009) Article ID 762478.
[6] L. Ciric, N. Cakid, M. Rajovic, J.S. Ume, Monotone generalized nonlinear contractions in partially ordered metric spaces, Fixed Point Theory Appl. (2008) Article ID 131294.
[7] T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379–1393.
[8] J. Harjani, K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal. 71 (2009) 3403–3410.
[9] J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal. 72 (2010) 1188–1197.
[10] V. Lakshmikantham, L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009) 4341–4349.
[11] J.J. Nieto, R. Rodríguez-López, Existence of extremal solutions for quadratic fuzzy equations, Fixed Point Theory Appl. (2005) 321–342.
[12] J.J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223–239.
[13] J.J. Nieto, R. Rodríguez-López, Applications of contractive-like mapping principles to fuzzy equations, Rev. Mat. Complut. 19 (2006) 361–383.
[14] J.J. Nieto, R.L. Pouso, R. Rodríguez-López, Fixed point theorems in ordered abstract spaces, Proc. Amer. Math. Soc. 135 (2007) 2505–2517.
[15] J.J. Nieto, R. Rodríguez-López, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Math. Sinica 23 (2007) 2205–2212.
[16] D. O'Regan, A. Petrusel, Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl. 341 (2008) 1241–1252.
[17] A. Petrusel, I.A. Rus, Fixed point theorems in ordered L-spaces, Proc. Amer. Math. Soc. 134 (2006) 411–418.
[18] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004) 1435–1443.
[19] Y. Wu, New fixed point theorems and applications of mixed monotone operator, J. Math. Anal. Appl. 341 (2008) 883–893.
[20] A. Cabada, J.J. Nieto, Fixed points and approximate solutions for nonlinear operator equations, J. Comput. Appl. Math. 113 (2000) 17–25.